

A Closed-Form Expression for the Maximum Inclination Angle of Build-and-Hold Directional Well Trajectories: A Unified Formulation Valid for Both $r_1 > X_3$ and $X_3 > r_1$

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ABSTRACT

The maximum (hold) inclination angle is the controlling design variable of a build-and-hold directional well, yet textbook solutions give two different expressions depending on whether the build radius exceeds the horizontal departure of the target, and the second expression is sensitive to quadrant errors. This study derives a single closed-form expression valid for both cases from the tangent geometry of the build arc, $\vartheta = 2 \tan^{-1}[X_3/((D_3 - D_1) + S)]$, where $S = [(D_3 - D_1)^2 + X_3^2 - 2r_1X_3]^{1/2}$ is the hold length. The same quantity yields a feasibility test and the minimum build rate needed to reach a given target. The expression reproduced three published worked examples to within 0.007° and closed the trajectory to within 5×10^{-12} ft in 193,120 randomly generated feasible designs. Parametric analysis showed that the maximum inclination rises by about 0.73° ($r_1 > X_3$) and 0.56° ($X_3 > r_1$) per 100 ft of horizontal departure, falls from 53.7° to 18.5° as target depth increases from 5,000 to 10,000 ft, increases from 16.9° to 48.3° as the kick-off point is deepened from 500 to 6,000 ft, and depends only weakly on build rate (20.8° to 18.7° for 1 to $5^\circ/100$ ft). A direct-difference simplification, $\vartheta = \sin^{-1}(X_3/A)$, explored in preliminary work was found to be inexact, with errors from -2.5° to $+15.7^\circ$ and no solution in a region where the exact solution exists; it is not recommended. The closed form gives well planners a transparent, branch-free calculation and a geometric kernel for trajectory-optimisation workflows.



Keywords: Directional drilling; build-and-hold profile; maximum inclination angle; well trajectory design; tangent geometry; closed-form solution

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INTRODUCTION

Directional drilling allows a wellbore to be steered along a planned path so that targets displaced laterally from the surface location can be reached. It increases the reservoir volume that can be accessed from one surface location, allows several wells to be drilled from a single pad and reduces the number of rig moves (Bourgoyne *et al.*, 1986; Samuel & Liu, 2009). Every directional well starts with a trajectory design in which the surface location, target coordinates, true vertical depth (TVD), kick-off point (KOP) and build rate are specified, and the inclination and azimuth that the bottom-hole assembly must achieve are calculated before drilling begins. Among the standard well profiles, the build-and-hold (slant) profile is one of the simplest and most widely used. The well is drilled vertically to the KOP, inclination is built at a constant rate to a maximum value θ at the end of build (EOB), and inclination and azimuth are then held constant along a straight tangent until the target is intersected. It is generally applied to single targets at moderate depth and, with a shallow KOP, to deeper wells with large horizontal departure.

The maximum inclination θ is the central design variable of this profile because it fixes the lengths of the build and hold sections and, through them, the dogleg, torque and drag demands on the drillstring. Textbook solutions obtain θ from the geometry of a circular arc and a tangent line, but they give two different expressions depending on whether the build radius r_1 is larger or smaller than the horizontal departure X_3 of the target (Samuel & Liu, 2009). The second expression nests a quadrant-sensitive arctangent inside an arccosine, so a calculator that returns principal values produces a wrong angle (demonstrated in the Results and Discussion). Wiggins and Juvkam-Wold (1990) showed that planar trajectories of several well types can be handled with a single governing equation; a similarly compact solution for the build-and-hold case that needs no branch logic would be useful to well planners. Survey calculation, in contrast to design, has converged on the minimum curvature method as the accepted industry standard (Sawaryn & Thorogood, 2005), and the planning-stage geometry of the build-and-hold profile rests on the same circular-arc idealisation. Over the past decade, research on trajectory design has concentrated on optimisation: metaheuristic and evolutionary algorithms have been applied to minimise wellbore length, torque and collision risk, or to trade these objectives against one another (Atashnezhad *et al.*, 2014; Guria *et al.*, 2014; Khosravianian *et al.*, 2018; Sha & Pan, 2018; Mansouri *et al.*, 2020; Yavari *et al.*, 2023; Li *et al.*, 2025). Such studies evaluate large numbers of candidate trajectories, each requiring the geometric quantities of the underlying profile. A closed-form, branch-free expression for θ is therefore a useful building block and also an independent check on optimiser output. The objectives of this paper are to: (i) derive a single closed-form expression for θ that is valid for $r_1 > X_3$ and $X_3 > r_1$ and that identifies when no solution

exists; (ii) verify it against three published worked examples and against a large set of random designs by direct reconstruction of the trajectory; (iii) quantify how θ responds to horizontal departure, TVD, build rate and KOP depth; and (iv) assess a direct-difference simplification, $\theta = \sin^{-1}(X_3/A)$, that was derived from the Build and Hold profile, and establish where it can and cannot be used.

METHODOLOGY

Problem definition and assumptions

The design problem is posed in the vertical plane that contains the surface location and the target (Figure 1). The origin is the surface location, X is horizontal departure and depth is measured downward. The well is vertical to the KOP at depth D_1 ; from the KOP, inclination is built at a constant rate q ($^\circ/100$ ft) along a circular arc of radius r_1 ; at the EOB the well is held at constant inclination θ along a straight tangent that intersects the target B at depth D_3 and departure X_3 . The following assumptions apply: (a) a planar profile with no azimuth turn; (b) constant build rate, so that the build section is a circular arc tangent to the vertical at the KOP; (c) a straight, constant-inclination hold section; and (d) deterministic inputs, so that survey uncertainty, bottom-hole-assembly tendencies and formation effects are not considered. Symbols are defined in (Table 1).

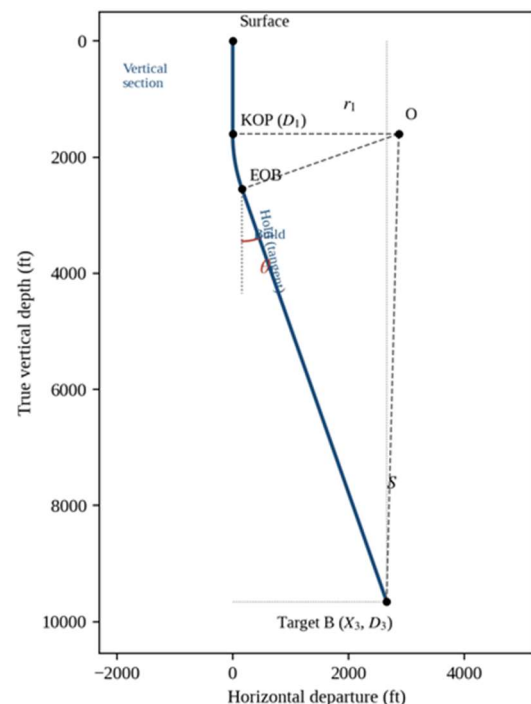


Figure 1: Geometry of a build-and-hold trajectory in the vertical plane (schematic, not to scale). The build arc has centre O and radius r_1 ; the hold section of length S is tangent to the arc at the EOB and passes through the target B, so that the segment OB has length A .

Table 1. Nomenclature.

Symbol	Description	Unit
D_1	TVD of the kick-off point (KOP)	ft
D_3	TVD of the target	ft
D	Vertical distance from KOP to target, $D_3 - D_1$	ft
X_3	Horizontal departure of the target	ft
q	Build rate (dogleg severity of the build section)	°/100 ft
r_1	Radius of curvature of the build section	ft
A	Distance from the build-arc centre O to the target	ft
S	Hold (tangent) length from EOB to target	ft
θ	Maximum (hold) inclination angle measured from the vertical	°
D_2, X_2	TVD and horizontal departure at the end of build (EOB)	ft
L_b	Length of the build section	ft
MD_T	Total measured depth to the target	ft
θ_s	Approximate inclination from the direct-difference expression, Eq. (16)	°
e_s	Signed error of the approximation, $\theta_s - \theta$	°
ξ, ρ	Dimensionless departure X_3/D and dimensionless radius r_1/D	—

Derivation of the governing relation

For a given build rate expressed in degrees per unit length, the radius of the build arc is

$$r_1 = \frac{180}{\pi} \left(\frac{1}{q} \right) \quad (1)$$

The arc centre is O = (r_1, D_1). A point C on the arc at which the inclination has reached θ has coordinates

$$x_C = r_1 (1 - \cos \theta) \quad , \quad z_C = D_1 + r_1 \sin \theta \quad (2)$$

and the well continues from C along the tangent direction ($\sin \theta, \cos \theta$) for a hold length S to reach the target:

$$X_3 = x_C + S \sin \theta \quad , \quad D_3 = z_C + S \cos \theta \quad (3)$$

Because the radius OC is perpendicular to the tangent, projecting the vector OB on the direction of OC gives the condition that the perpendicular distance from O to the tangent line equals r_1 . Eliminating S from Eqs. (2) and (3) yields

$$(r_1 - X_3) \cos \theta + (D_3 - D_1) \sin \theta = r_1 \quad (4)$$

With A defined by

$$A = \sqrt{(D_3 - D_1)^2 + (r_1 - X_3)^2} \quad (5)$$

Eq. (4) can be written as $A \sin(\theta + \tau) = r_1$, where $\tau = \tan^{-1}[(r_1 - X_3)/(D_3 - D_1)]$ is evaluated with the quadrant-aware arctangent (atan2). Taking the principal branch gives the angle-difference form

$$\theta = \sin^{-1} \left(\frac{r_1}{A} \right) - \tan^{-1} \left[\frac{(r_1 - X_3)}{(D_3 - D_1)} \right] \quad (6)$$

which is the geometric relation $\theta = \Omega - \tau$ with $\Omega = \sin^{-1}(r_1/A)$. It holds for both $r_1 > X_3$ (where $\tau > 0$) and $X_3 > r_1$ (where $\tau < 0$) without branching, provided τ is evaluated with atan2.

Single-arctangent form, hold length and feasibility

A form that needs only one inverse function and avoids the quadrant bookkeeping follows from the half-angle substitution $t = \tan(\theta/2)$, with $\cos \theta = (1 - t^2)/(1 + t^2)$ and $\sin \theta = 2t/(1 + t^2)$. Substituting into Eq. (4) gives the quadratic

$$(2r_1 - X_3)t^2 - 2(D_3 - D_1)t + X_3 = 0 \quad (7)$$

whose discriminant is the square of the hold length. Because $|OB|^2 = r_1^2 + S^2$ for a tangent from B to the build circle,

$$S = \sqrt{(D_3 - D_1)^2 + X_3^2 - 2r_1 X_3} \quad (8)$$

The roots of Eq. (7) are $t = [(D_3 - D_1) \pm S]/(2r_1 - X_3)$. The smaller root is physically meaningful; the larger root gives an inclination above 90° and corresponds to the second tangent from B to the circle, which touches the far side of the build circle and is not a build-and-hold path. Rationalising the smaller root removes the division by $(2r_1 - X_3)$ and gives the main result of this paper:

$$\theta = 2 \tan^{-1} \left[\frac{X_3}{(D_3 - D_1) + S} \right] \quad (9)$$

Equation (9) contains one inverse trigonometric function, one square root and no branch condition, and it is valid for any ratio of r_1 to X_3 , including $X_3 > 2r_1$. A real tangent exists only when the radicand in Eq. (8) is non-negative, that is, when the target lies on or outside the build circle:

$$(D_3 - D_1)^2 + X_3^2 \geq 2r_1 X_3 \quad (10)$$

Rearranging Eq. (10) with Eq. (1) gives the minimum build rate (per unit length, multiplied by 100 for °/100 ft) required to reach the target:

$$q_{min} = \frac{180}{\pi} \frac{2X_3}{(D_3 - D_1)^2 + X_3^2} \quad (11)$$

The remaining design quantities follow directly. The build length and total measured depth are

$$L_b = \frac{\pi}{180} r_1 \theta, \quad MD_T = D_1 + L_b + S \quad (12)$$

and the EOB coordinates are obtained by setting θ in Eq. (2): $D_2 = D_1 + r_1 \sin \theta$ and $X_2 = r_1(1 - \cos \theta)$. Two limits provide consistency checks. When the build radius vanishes (an instantaneous kick), Eq. (9) reduces to the straight-line slant angle,

$$\theta \rightarrow \tan^{-1} \left(\frac{X_3}{D_3 - D_1} \right) \quad (13)$$

and the finite build radius therefore adds a “build penalty” of $\theta - \tan^{-1}(X_3/D)$ to the straight-line angle. Introducing $\xi = X_3/D$ and $\rho = r_1/D$ collapses all designs onto a single two-parameter relation,

$$\theta = 2 \tan^{-1} \left[\frac{\xi}{1 + \sqrt{1 + \xi^2 - 2\rho\xi}} \right] \quad (14)$$

Reference formulations used for comparison

Two formulations were used as comparators. The first is the textbook two-branch solution (Samuel & Liu, 2009): for $r_1 > X_3$ it coincides with Eq. (6), while for $X_3 > r_1$ it is

$$\theta = \tan^{-1} \left[\frac{(D_3 - D_1)}{r_1 - X_3} \right] - \cos^{-1} \left\{ \frac{r_1}{D_3 - D_1} \sin \tan^{-1} \left[\frac{(D_3 - D_1)}{r_1 - X_3} \right] \right\} \quad (15)$$

and requires the arctangent to be evaluated in the second quadrant (90° to 180°). The second comparator is a direct-difference simplification of Eq. (6) examined in preliminary work. If $\Omega = \sin^{-1}(r_1/A)$ and $\tau = \sin^{-1}((r_1 - X_3)/A)$ are subtracted “as if” $\sin^{-1}\{a\} - \sin^{-1}\{b\} = \sin^{-1}\{a - b\}$, the result is

$$\theta_s = \sin^{-1} \left(\frac{X_3}{A} \right) \quad (16)$$

This replacement is not an identity. The exact relation, obtained from $\sin(\Omega - \tau)$ with $\cos \Omega = S/A$ and $\cos \tau = (D_3 - D_1)/A$, is

$$\sin \theta = \frac{r_1(D_3 - D_1) - (r_1 - X_3)S}{A^2} \quad (17)$$

so θ_s in Eq. (16) can match θ only by coincidence. The signed error of the approximation is defined as

$$e_s = \theta_s - \theta \quad (18)$$

Verification, numerical experiments and implementation

Verification followed four steps. First, Eq. (9) was applied to three worked examples from the drilling literature (Table 2), using the rounded build radii printed in those sources so that results are directly comparable with the published angles. Second, a closure test was run on random designs: build rate $q \sim U(0.5, 6)^\circ/100$ ft, KOP depth $D_1 \sim U(500, 5,000)$ ft, vertical distance $D \sim U(3,000, 15,000)$ ft and departure $X_3 \sim U(200, 8,000)$ ft. For each feasible design (Eq. 10) the angle was computed from Eq. (9), the EOB was located with Eq. (2), the well was extended along the tangent for the hold length S , and the distance between the endpoint and the target was recorded as the closure residual. Eq. (9) was also compared with Eq. (6) and with the quadrant-corrected Eq. (15). Third, parametric studies were run around the worked-example inputs: horizontal departure X_3 (Example 1 inputs for $r_1 > X_3$; Example 2 inputs for $X_3 > r_1$), target TVD D_3 at fixed X_3 , build rate q and KOP depth D_1 . Fourth, the approximation of Eq. (16) was scored with Eq. (18) on the same grids. All calculations used double-precision arithmetic in Python (NumPy), with plots produced in Matplotlib. Angles are in degrees and lengths in feet, consistent with the source examples.

Table 2: Inputs for the three published worked examples used for validation.

Example	Source	q ($^\circ/100$ ft)	D_1 (ft)	D_3 (ft)	X_3 (ft)	r_1 (ft)	Case
1	Applied drilling Engineering te(SPE Textbook Series, 1986)	2.0	1,600	9,650	2,655	2,865	$r_1 > X_3$
2	Tabatabaei Nejad, S.A & Owangi. M.(2011)	1.5	2,500	12,500	4,500	3,820	$X_3 > r_1$
3	Heriot-Watt drilling text, p. 478 (Brown, 2000)	2.0	2,000	10,000	3,000	2,865	$X_3 > r_1$

r_1 values are those printed in the sources (rounded). Exact conversion by Eq. (1) gives 2,864.8 ft ($q = 2^\circ/100$ ft) and 3,819.7 ft ($q = 1.5^\circ/100$ ft).

RESULTS AND DISCUSSION

Validation against published worked examples

Equation (9) reproduced the published inclination angle in all three worked examples (Table 3). The deviation from the published value was 0.007° , 0.003° and -0.001° for

Examples 1 to 3, which is within the rounding of the published angles and build radii. Because the textbook formulae and Eq. (9) derive from the same tangent geometry, agreement is expected; the practical gain is that Eq. (9) is applied unchanged to Example 1 ($r_1 > X_3$) and to Examples 2 and 3 ($X_3 > r_1$). By contrast, the textbook second-branch expression of Eq. (15) returned -198.5°

Table 3: Maximum inclination angle θ for the worked examples: published value, closed form (Eq. 9) and direct-difference approximation (Eq. 16).

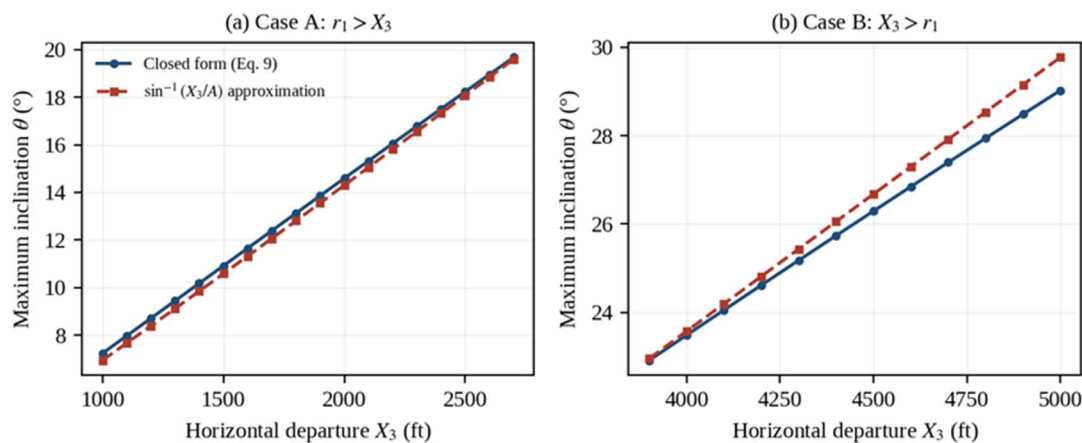
Example	Published θ (°)	Eq. (9) θ (°)	Deviation, Eq. (9) – published (°)	Approximation θ_s (°)	Error e_s (°)
1	19.34	19.347	0.007	19.251	-0.096
2	26.29	26.293	0.003	26.677	0.384
3	21.95	21.949	-0.001	22.021	0.072

Error e_s is measured against Eq. (9). The approximation is evaluated with the same rounded r_1 as the sources.

Table 4. Derived design quantities for the worked examples computed with Eqs. (2), (8), (12) and (13).

Example	θ (°)	D_2 (ft)	X_2 (ft)	L_b (ft)	S (ft)	MD_T (ft)	Build penalty (°)
1	19.35	2,549	162	967	7,526	10,093	1.09
2	26.29	4,192	395	1,753	9,267	13,520	2.07
3	21.95	3,071	208	1,098	7,471	10,568	1.39

Build penalty = $\theta - \tan^{-1}(X_3/D)$.

**Figure 2.** Maximum inclination θ versus horizontal departure X_3 : (a) Case A, $r_1 > X_3$ ($D_1 = 1,600$ ft, $D_3 = 9,650$ ft, $r_1 = 2,865$ ft); (b) Case B, $X_3 > r_1$ ($D_1 = 2,500$ ft, $D_3 = 12,500$ ft, $r_1 = 3,820$ ft). The dashed curve is the direct-difference approximation, Eq. (16).

and -200.0° for Examples 2 and 3 when the arctangent was evaluated as a principal value, and gave the correct 26.29° and 21.95° only after the quadrant was corrected manually. This confirms that the second branch is error-prone in routine hand or spreadsheet use. Table 4 lists the derived design quantities for the three examples, which are not reported in the sources and are given for completeness. The finite build radius raises the hold inclination by 1.1° , 2.1° and 1.4° above the straight-line angle $\tan^{-1}(X_3/D)$, and the build section accounts for about 10% to 13% of the total measured depth.

Numerical closure and consistency

Of the 200,000 random designs generated, 193,120 were feasible. For all of them, the closure residual, the distance between the end of the reconstructed trajectory and the target, did not exceed 4.5×10^{-12} ft, and the largest differences between Eq. (9) and both Eq. (6) and the quadrant-corrected Eq. (15) were 3.1×10^{-13} degrees. These values are at the level of double-precision round-off. As the build radius was reduced towards zero, Eq. (9)

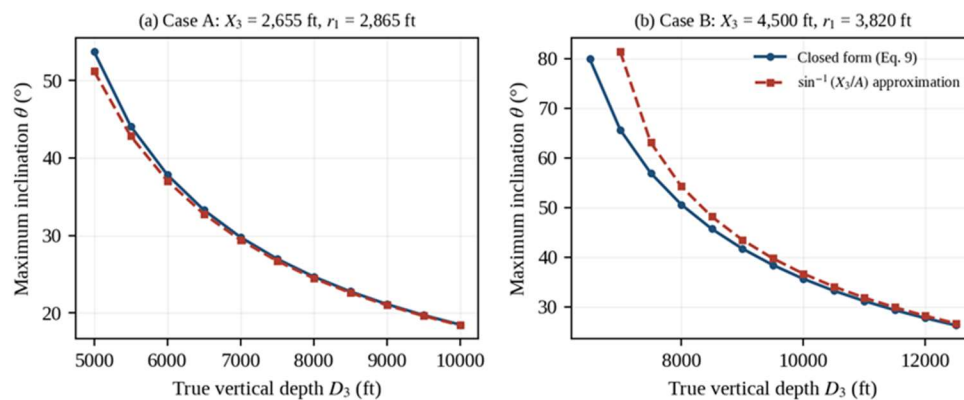
converged to Eq. (13): for $D = 8,000$ ft and $X_3 = 3,000$ ft it returned 20.556045° against $\tan^{-1}(3,000/8,000) = 20.556045^\circ$. Equation (9) therefore reproduces the exact tangent geometry over the full range tested, and it does so without selecting a branch.

Effect of horizontal departure

Figure 2 and Table 5 show the variation of θ with the horizontal departure X_3 . In Case A (Example 1 inputs, $r_1 = 2,865$ ft), increasing X_3 from 1,000 to 2,700 ft raised θ from 7.2° to 19.7° , an average of 0.73° per 100 ft. In Case B (Example 2 inputs, $r_1 = 3,820$ ft), increasing X_3 from 3,900 to 5,000 ft raised θ from 22.9° to 29.0° , an average of 0.56° per 100 ft. The relation is monotonic and close to linear over these ranges in both cases, and the sensitivity is lower in Case B. A planner can therefore expect every additional 100 ft of departure to cost roughly 0.6° to 0.7° of hold inclination at these depths, which must be set against the casing, torque-and-drag and hole-cleaning limits of the section.

Table 5: Effect of horizontal departure X_3 on the maximum inclination angle.

X_3 (ft)	θ , Eq. (9) (°)	θ_s , Eq. (16) (°)	Error e_s (°)
Case A: $r_1 > X_3$ ($D_1 = 1,600$ ft, $D_3 = 9,650$ ft, $r_1 = 2,865$ ft)			
1,000	7.243	6.951	-0.292
1,300	9.447	9.121	-0.325
1,600	11.654	11.323	-0.330
1,900	13.858	13.553	-0.305
2,200	16.052	15.805	-0.247
2,500	18.230	18.074	-0.156
2,700	19.670	19.593	-0.077
Case B: $X_3 > r_1$ ($D_1 = 2,500$ ft, $D_3 = 12,500$ ft, $r_1 = 3,820$ ft)			
3,900	22.915	22.954	0.039
4,100	24.052	24.195	0.143
4,300	25.178	25.436	0.258
4,500	26.293	26.677	0.384
4,700	27.396	27.917	0.521
5,000	29.024	29.772	0.748

**Figure 3.** Maximum inclination θ versus target depth D_3 : (a) Case A with $X_3 = 2,655$ ft, $D_1 = 1,600$ ft, $r_1 = 2,865$ ft; (b) Case B with $X_3 = 4,500$ ft, $D_1 = 2,500$ ft, $r_1 = 3,820$ ft. The dashed curve is the direct-difference approximation, Eq. (16); no solution exists for $D_3 = 6,000$ ft in Case B.**Table 6.** Effect of target depth D_3 on the maximum inclination angle.

D_3 (ft)	θ , Eq. (9) (°)	θ_s , Eq. (16) (°)	Error e_s (°)
Case A: $X_3 = 2,655$ ft, $D_1 = 1,600$ ft, $r_1 = 2,865$ ft			
5,000	53.716	51.205	-2.511
6,000	37.839	37.065	-0.774
7,000	29.789	29.426	-0.363
8,000	24.699	24.495	-0.203
9,000	21.143	21.017	-0.126
10,000	18.504	18.420	-0.084
Case B: $X_3 = 4,500$ ft, $D_1 = 2,500$ ft, $r_1 = 3,820$ ft			
6,000	No solution ($S^2 < 0$)	undefined	—
6,500	79.952	undefined	—
7,000	65.666	81.407	15.741
8,000	50.623	54.292	3.669
9,000	41.740	43.515	1.775
10,000	35.662	36.695	1.033
11,000	31.188	31.852	0.664
12,000	27.740	28.195	0.455
12,500	26.293	26.677	0.384

Effect of target depth

Figure 3 and Table 6 show that θ decreases as the target is drilled deeper for a fixed departure. In Case A,

increasing D_3 from 5,000 to 10,000 ft reduced θ from 53.7° to 18.5°, with most of the reduction occurring below 7,000 ft: the first 2,000 ft of added depth lowered the angle by 23.9°, whereas the last 3,000 ft lowered it by 11.3°.

A deeper target gives the hold section more vertical room to cover the same departure at a smaller angle. Case B exposes the behaviour near the limit of feasibility. At $D_3 = 12,500$ ft the angle is 26.3° , but it rises steeply to 65.7° at 7,000 ft and 80.0° at 6,500 ft, and at 6,000 ft no tangent exists because the target lies inside the build circle ($S^2 < 0$). Equation (11) quantifies this: at $D_3 = 6,000$ ft a build rate of at least $1.59^\circ/100$ ft is needed, so the $1.5^\circ/100$ ft assumed in the example cannot reach the target, whereas at 6,500 ft the minimum falls to $1.42^\circ/100$ ft and the design becomes feasible, albeit with an inclination close to 80° . Designs that approach this boundary give a very steep hold section for a small change in input and should be avoided; the feasibility test and $q_{\min}$ of Eq. (11) make this check explicit.

Effect of build rate and kick-off depth

The build rate has a modest effect on the hold inclination (Table 7). For Example 1 inputs, raising q from 1 to $5^\circ/100$ ft reduced θ from 20.8° to 18.7° and shortened the total measured depth by only 31 ft (0.3%), because the angle approaches the straight-line limit of 18.3° as r_1 decreases. The build rate therefore matters mainly through dogleg and torque limits rather than through the target-hitting geometry. The KOP depth has a much larger effect (Table 8): deepening the KOP from 500 to 6,000 ft raised θ from 16.9° to 48.3° and lengthened the well by 648 ft, since less vertical distance remains for the build and hold sections to cover the same departure.

Table 7: Effect of build rate q on the hold inclination and well length ($D_1 = 1,600$ ft, $D_3 = 9,650$ ft, $X_3 = 2,655$ ft).

q ($^\circ/100$ ft)	r_1 (ft)	θ ($^\circ$)	L_b (ft)	S (ft)	MD_T (ft)
1.0	5,730	20.77	2,077	6,436	10,114
1.5	3,820	19.78	1,318	7,181	10,100
2.0	2,865	19.35	967	7,526	10,093
3.0	1,910	18.95	632	7,856	10,087
4.0	1,432	18.77	469	8,015	10,085
5.0	1,146	18.66	373	8,110	10,083

Table 8: Effect of KOP depth D_1 on the hold inclination and well length ($D_3 = 9,650$ ft, $X_3 = 2,655$ ft, $r_1 = 2,865$ ft).

D_1 (ft)	θ ($^\circ$)	L_b (ft)	S (ft)	MD_T (ft)
500	16.93	846	8,692	10,039
1,000	17.95	897	8,164	10,062
1,600	19.35	967	7,526	10,093
2,500	21.93	1,097	6,554	10,151
4,000	28.32	1,416	4,874	10,290
6,000	48.30	2,415	2,271	10,686

A shallow KOP lowers the inclination, but it moves the build section into shallower, often softer formations and may conflict with casing-point and collision constraints that are outside the scope of this geometric model.

Dimensionless design map

Figure 4 maps Eq. (14) over the dimensionless departure ξ and radius ρ . The three worked examples lie in a narrow band ($\rho = 0.36$ to 0.38 ; $\xi = 0.33$ to 0.45), where the contours are nearly vertical: θ is governed mainly by the departure-to-depth ratio and only secondarily by the build radius, consistent with the build penalties of 1° to 2° in Table 4. The contours bend away from vertical as ρ approaches the infeasible region, where the angle rises rapidly, in line with the Case B behaviour in Section 3.4. The map allows a quick feasibility and sensitivity check for any combination of depth, departure and build rate before detailed planning.

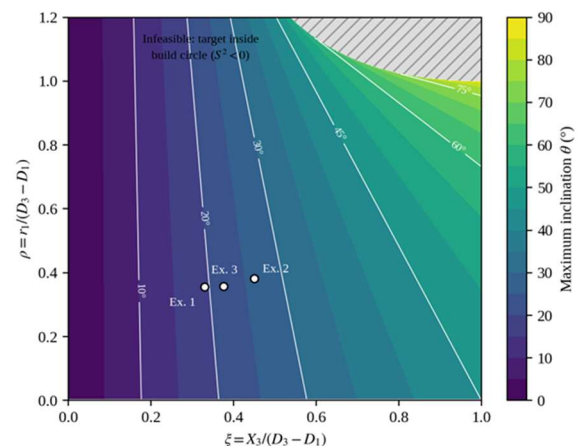


Figure 4. Dimensionless map of the maximum inclination angle from Eq. (14). The hatched region is infeasible ($S^2 < 0$). Circles mark the three worked examples.

Performance of the direct-difference approximation

Across the 53 parametric cases for which both quantities exist, the approximation θ_s of Eq. (16) had errors from -2.51° to $+15.74^\circ$ with a mean absolute error of 0.95° . Errors were small and negative in Case A for the departure sweep (-0.33° to -0.08°) and positive in Case B ($+0.04^\circ$ to $+0.75^\circ$), but they became large wherever the target was shallow relative to its departure: -2.51° at $D_3 = 5,000$ ft in Case A and $+15.74^\circ$ at $D_3 = 7,000$ ft in Case B (Table 6; Figure 3). In Case B the error stayed above 0.5° until D_3 reached 12,000 ft. The domain of θ_s also differs from the true feasibility domain: θ_s requires $X_3 \leq A$, whereas a tangent exists when $A \geq r_1$. At $D_3 = 6,500$ ft in Case B, $A = 4,057$ ft lies between $r_1 = 3,820$ ft and $X_3 = 4,500$ ft, so the exact angle is 79.95° but θ_s is undefined. The small errors in the published-example comparison (Table 3: -0.10° , $+0.38^\circ$ and $+0.07^\circ$) therefore reflect operation far from these limits and cannot be extrapolated. Because Eq. (9) is no more laborious than Eq. (16) and is exact within the model, the approximation offers no advantage and should not be used for design. Tabulated results and trends that were previously reported with Eq. (16) should be recomputed with Eq. (9), as in (Tables 5 and 6).

Practical implications and limitations

Equation (9), with Eqs. (8), (10) and (12), gives a complete planar build-and-hold design in a few lines of code or a spreadsheet: feasibility, hold inclination, EOB position, build and hold lengths and total measured depth, with no case selection. It can serve as an analytical kernel inside the optimisation frameworks cited above, where geometric feasibility and hold angle must be evaluated for each candidate KOP and build-rate pair, and as an independent check of commercial planning output. In such frameworks the objective functions of length, torque and friction, collision separation and risk (Mansouri *et al.*, 2020; Yavari *et al.*, 2023; Li *et al.*, 2025) can be evaluated on a profile whose geometry is known exactly.

The solution is planar, so targets that require an azimuth change, and wells with a turn in the build section, need a three-dimensional treatment; the minimum curvature method (Sawaryn & Thorogood, 2005) is the appropriate tool for the corresponding survey calculations. The build rate is assumed constant and achieved exactly, whereas real assemblies deliver a variable dogleg and are subject to tool-face and formation effects. Survey uncertainty, anti-collision requirements and wellbore stability, which govern the acceptable range of θ , are not part of the geometry. The numerical results are for the three published examples and for parameter ranges based on them; they illustrate trends rather than establish field-wide rules, and no field data were used. Extending the closed form to build-hold-drop (S-type) profiles, in which the upper section is identical to the profile treated here, and to three-dimensional profiles is a natural next step.

Conclusion

A single closed-form expression, $\theta = 2 \tan^{-1}[X_3/((D_3 - D_1) + S)]$ with $S = [(D_3 - D_1)^2 + X_3^2 - 2r_1X_3]^{1/2}$, was derived for the maximum inclination angle of a build-and-hold trajectory. The main findings are:

1. The expression is valid for $r_1 > X_3$ and $X_3 > r_1$ alike, needs no branch selection or quadrant correction, and returns the hold length S , a feasibility test (Eq. 10) and the minimum build rate (Eq. 11) from the same quantities.

2. It reproduced three published worked examples to within 0.007° and closed the reconstructed trajectory to within 5×10^{-12} ft in 193,120 random feasible designs. The textbook second-branch formula, evaluated with principal-value arctangents, returned angles near -200° for the two $X_3 > r_1$ examples.

3. For the example geometries, the maximum inclination increased by about 0.73° (Case A) and 0.56° (Case B) per 100 ft of horizontal departure and decreased from 53.7° to 18.5° as target depth increased from 5,000 to 10,000 ft. It rose sharply near the feasibility limit (79.95° at the shallowest feasible depth tested in Case B). Deepening the KOP from 500 to 6,000 ft raised it from 16.9° to 48.3° , whereas raising the build rate from 1 to 5°/100 ft lowered it only from 20.8° to 18.7° .

4. The direct-difference simplification $\sin^{-1}(X_3/A)$ is not an identity of the geometry. Its errors ranged from -2.5° to $+15.7^\circ$ over the cases examined, and it was undefined where the exact solution exists. It should not be used for design.

5. The closed form is limited to planar, constant-build profiles. It can be used directly in planning calculations and as a geometric kernel in trajectory-optimisation studies; extension to three-dimensional and build-hold-drop profiles is recommended for future work.

Declarations

Conflict of interest: The author declares no competing interests. **Funding:** This research received no external funding.

Data availability: All inputs and results needed to reproduce the calculations are given in the tables, and Eqs. (1), (8), (9), (10) and (12) are sufficient to implement the method.

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