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**Research Article**  
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## A Class of Hybrid Multi-derivative linear multistep methods for direct solving third-order ordinary differential equations

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### ABSTRACT

This article considered a Class of Hybrid Multi-derivative Linear Multistep Method (CHMLMM) for solving third-order ordinary differential equations. Power series was adopted as the basis function to derive the method. In order to get a desired solution, the basis function was approximated and interpolated at some selected off-grid points while the third derivative of the approximate solution was collocated at all grid and off-grid points to generate a system of linear equations for the determination of the unknown parameters. The basic properties of the method were examined. The derived method was implemented in block mode to solve third-order ordinary differential equations to demonstrate the usability and efficiency of the methods. The absolute error obtained in the numerical experiments showed a better performance of the present method over some of the existing methods in the literature.

**Keywords:** Ordinary differential equation, third-order, hybrid, interpolation and collocation, boundary layer

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**INTRODUCTION**

The attempt to find better solution to Ordinary Differential Equations have led researchers to developing several methods over the years, with each coming out with varying degree of solutions. While some of these solutions are easy to relate with, some requires further interpretations to get the ideas behind them, thus the numerical solutions to such equations comes handy in given expression to the basic interpretation to what a solution give when compared with the exact solution.

This article discussed the approximate solution to general third-order ordinary differential equations using the Class of Hybrid Multi-derivative Linear Multistep Method (CHMLMM). Thus, we seek to find numerical solution to equation of the form:

$$y'''(x) = f(x, y(x), y'(x), y''(x)), \quad y(a) = y_0, y'(a) = \delta_0, y''(a) = y_N \tag{1}$$

where  $a, y_0, \delta_0, y_N$ , and  $f$  is a continuous function and satisfies a Lipschitz condition as given in [1].

The solution of (1) is conventionally solved by first reducing it to equivalent first-order differential equation, which have been considered by authors [1], [2], [3] and others. In like manner, authors such as [4], [5], to mention a few, have developed different method to solving equation (1) with each coming out with various degree of results.

In recent times, authors ([4], [5], [6]) have made tremendous progress in developing methods for the solution of (1) without reduction to equivalent first-order ordinary differential equation with each coming out with varying degree of successes in their approach. Many other numerical approaches has been considered by ([7], [8], [9], [10], [11], [12], [13], [14], [15])

While the search for more accurate numerical methods continues. This article develops a Class of Hybrid Multi-derivative Linear Multistep Method (CHMLMM), implemented in block mode, for directly solution of equation (1). The article is sectioned as follows: Section 1 is an introduction, Section 2 discusses how the methods were derived, and Section 3 analyses the method's basic properties. Section 4 gives numerical results to show the advantages of speed and accuracy. Section 5 provides a summary and conclusion.

**Derivation of the Block Method**

The solution of (1) is considered in the interval [0,5] by allowing  $y(x)$  to be approximated by partial sum of power series polynomial of the form

$$y(x) \approx p(x) = \sum_{j=0}^{20} a_j x^j \tag{2}$$

where  $x$  is continuously differentiable and  $a_j$ 's are parameters to be determine. In order to apply the procedure of collocation and interpolation to (2),third, fourth and fifth derivatives are obtained as:

$$y'''(x) \approx p'''(x) = \sum_{j=3}^{20} j(j-1)(j-2)a_j x^{j-3}, \tag{3}$$

$$y^{(4)}(x) = \sum_{j=4}^{20} j(j-1)(j-2)(j-3)a_j x^{j-4}, \tag{4}$$

and

$$y^{(5)}(x) = \sum_{j=5}^{20} j(j-1)(j-2)(j-3)(j-4)a_j x^{j-5}. \tag{5}$$

respectively. Note that the following are equivalent;

$$y'''(x, y(x), y'(x), y''(x)) \approx f(x, y(x), y'(x), y''(x)), \quad y^{(4)}(x, y(x), y'(x), y''(x)) \approx g(x, y(x), y'(x), y''(x)), \quad \text{and} \\ y^{(5)}(x, y(x), y'(x), y''(x)) \approx \gamma(x, y(x), y'(x), y''(x)). \text{ Interpolating (2) at } x = x_{n+j}, j = 0, 1, \frac{9}{2} \text{ and collocating (3) to (5) at } x = x_{n+j}, j = 0, 1, 2, 3, 4, \text{ and } 5 \text{ yield the system of equations that is written in matrix form as}$$

$$XA = B \tag{6}$$

where,

$$X = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & \dots & x_n^{20} \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & \dots & x_{n+1}^{20} \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & \dots & x_{n+2}^{20} \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 & \dots & 4080x_n^{17} \\ 0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 & \dots & 6840x_{n+1}^{17} \\ 0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & 210x_{n+2}^4 & \dots & 6840x_{n+2}^{17} \\ 0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & 210x_{n+3}^4 & \dots & 6840x_{n+3}^{14} \\ 0 & 0 & 0 & 6 & 24x_{n+4} & 60x_{n+4}^2 & 120x_{n+4}^3 & 210x_{n+4}^4 & \dots & 6840x_{n+4}^{17} \\ 0 & 0 & 0 & 6 & 24x_{n+5} & 60x_{n+5}^2 & 120x_{n+5}^3 & 210x_{n+5}^4 & \dots & 6840x_{n+5}^{17} \\ 0 & 0 & 0 & 0 & 24 & 60x_n & 120x_n^2 & 360x_n^3 & \dots & 116280x_n^{16} \\ 0 & 0 & 0 & 0 & 24 & 60x_{n+1} & 120x_{n+1}^2 & 360x_{n+1}^3 & \dots & 116280x_{n+1}^{16} \\ 0 & 0 & 0 & 0 & 24 & 60x_{n+2} & 120x_{n+2}^2 & 360x_{n+2}^3 & \dots & 116280x_{n+2}^{16} \\ 0 & 0 & 0 & 0 & 24 & 60x_{n+3} & 120x_{n+3}^2 & 360x_{n+3}^3 & \dots & 116280x_{n+3}^{16} \\ 0 & 0 & 0 & 0 & 24 & 60x_{n+4} & 120x_{n+4}^2 & 360x_{n+4}^3 & \dots & 116280x_{n+4}^{16} \\ 0 & 0 & 0 & 0 & 24 & 60x_{n+5} & 120x_{n+5}^2 & 360x_{n+5}^3 & \dots & 116280x_{n+5}^{16} \\ 0 & 0 & 0 & 0 & 0 & 120 & 720x_n & 2520x_n^2 & \dots & 1860480x_n^{15} \\ 0 & 0 & 0 & 0 & 0 & 120 & 120x_{n+1} & 720x_{n+1}^2 & \dots & 1860480x_{n+1}^{15} \\ 0 & 0 & 0 & 0 & 0 & 120 & 120x_{n+2} & 720x_{n+2}^2 & \dots & 1860480x_{n+2}^{15} \\ 0 & 0 & 0 & 0 & 0 & 120 & 120x_{n+3} & 720x_{n+3}^2 & \dots & 1860480x_{n+3}^{15} \\ 0 & 0 & 0 & 0 & 0 & 120 & 120x_{n+4} & 720x_{n+4}^2 & \dots & 1860480x_{n+4}^{15} \\ 0 & 0 & 0 & 0 & 0 & 120 & 120x_{n+5} & 720x_{n+5}^2 & \dots & 1860480x_{n+5}^{15} \end{pmatrix}$$

$$B = (y_n, y_{n+1}, y_{n+\frac{9}{2}}, f_n, f_{n+1}, f_{n+2}, \dots, f_{n+5}, g_n, g_{n+1}, g_{n+2}, \dots, g_{n+5}, \gamma_n, \gamma_{n+1}, \gamma_{n+2}, \dots, \gamma_{n+5})$$

$A = (a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, \dots, a_{20})$ . Equation (6) is solved for  $a_j$ 's using Gaussian elimination methods and substitute into equation (2), which yields after simple simplification the continuous scheme below

$$y(n) = \alpha_0 y_n + \alpha_1 y_{n+1} + \frac{\alpha_2 y_{n+\frac{9}{2}}}{2} + h^3 \sum_{j=0}^3 \beta_j f_{n+j} + h^4 \sum_{j=0}^3 \mu_j g_{n+j} + h^5 \sum_{j=0}^3 \nu_j \gamma_{n+j} \quad (7)$$

where  $\alpha_j, \beta_j, \mu_j,$  and  $\nu_j$  are the coefficients that define the scheme. The coefficients are presented in the appendix. Evaluating (7) and its first and second derivatives at  $n = 1, 2, 3, 4, 5$  yields the proposed Class of Hybrid Multi-derivative Linear Multistep Method (CHMLMM) whose formulas are as presented in equations (8), (9), (10).

$$\begin{aligned} & y_3 + \frac{2}{3} y_0 - \frac{9}{7} y_1 - \frac{8}{21} y_2^9 = * 1mm \\ & + h^5 \left( -\frac{1248419368255}{61667543089152} \gamma_3 - \frac{4059173}{164282499072} \gamma_4 - \frac{88939210351}{20555847696384000} \gamma_5 - \frac{3980464170227}{61667543089152000} \gamma_0 - \frac{12826048381}{1149977493504} \gamma_1 - \frac{92}{2055} \right) \\ & + h^4 \left( \frac{139819653739}{2854978846720} g_3 + \frac{71734478327}{29899414831104} g_4 + \frac{1789075812221}{29365496709120000} g_5 - \frac{768034199196479}{616675430891520000} g_0 - \frac{6761074520173}{149497074155520} g_1 + \frac{2}{2} \right) \\ & + h^3 \left( -\frac{349539249095891}{616675430891520} f_2 - \frac{136270262935009}{264289470382080} f_3 - \frac{10315051528291}{274077969285120} f_4 - \frac{32277834242192507}{4625065731686400000} f_0 - \frac{102156789920981}{274077969285120} f_1 - \right) \end{aligned} \quad (8)$$

$$\begin{aligned} & y_4 + \frac{1}{3} y_0 - \frac{4}{7} y_1 - \frac{16}{21} y_2^9 = * 1mm \\ & + h^5 \left( -\frac{484605152575}{30833771544576} \gamma_3 - \frac{1713681161}{1724966240256} \gamma_4 - \frac{14615357039}{10277923848192000} \gamma_5 - \frac{980898227443}{30833771544576000} \gamma_0 - \frac{3260151677}{574988746752} \gamma_1 - \frac{22}{1027} \right) \\ & + h^4 \left( \frac{432168473183}{23125328658432} g_3 + \frac{6660410920043}{672736833699840} g_4 + \frac{6280249812289}{308337715445760000} g_5 - \frac{569907935377853}{925013146337280000} g_0 - \frac{15738062336981}{672736833699840} g_1 \right) \\ & + h^3 \left( -\frac{31916933928689}{102779238481920} f_2 - \frac{54581475515873}{132144735191040} f_3 - \frac{310430491531889}{3700052585349120} f_4 - \frac{8009785256965243}{2312532865843200000} f_0 - \frac{699214414615159}{3700052585349120} f_1 \right) \end{aligned} \quad (9)$$

$$\begin{aligned}
 & y_5 - \frac{4}{9} y_0 + \frac{5}{7} y_1 - \frac{80}{63} y_2 = * 1mm \\
 & + h^5 \left( \frac{266597499451}{92501314633728} Y_3 + \frac{71941521619}{5174898720768} Y_4 + \frac{270119750389}{2642894703820800} Y_5 + \frac{105976232933}{205584769638400} Y_0 + \frac{31607652269}{5174898720768} Y_1 + \frac{18225}{92501} \right. \\
 & + h^4 \left( \frac{34484771803}{69375985975296} g_3 + \frac{5853458785927}{134547366739968} g_4 + \frac{119853360804757}{61667543089152000} g_5 + \frac{527613061301827}{555007887802368000} g_0 + \frac{314174997809}{14949707415552} g_1 + \right. \\
 & \left. + h^3 \left( \frac{332769179136391}{555007887802368} f_2 + \frac{79286841114624}{9286841114624} f_3 + \frac{885364378117519}{2220031551209472} f_4 + \frac{7097325818874757}{1387519719505920000} f_0 + \frac{496590455034249}{2220031551209472} f_1 + \right. \right. \\
 & \left. \left. (10) \right) \right)
 \end{aligned}$$

and additional schemes

$$\begin{aligned}
 & hy'_0 + \frac{11}{9} y_0 + \frac{4}{63} y_2 - \frac{9}{7} y_1 = \\
 & + h^5 \left( \frac{1180929154139503}{31244888498503680} Y_1 + \frac{141277544615717}{334766662483968000} Y_0 + \frac{25235404015807}{781122212462592000} Y_5 \right. \\
 & \left. - \frac{341994464711609}{3905611062312960} Y_2 + \frac{7628820288593}{128756408647680} Y_3 - \frac{12112857990379}{2403452961423360} Y_4 \right) \\
 & + h^4 \left( \frac{921701851519}{11487091359744} g_2 - \frac{17704343622139}{108489196175360} g_3 - \frac{3566597767865339}{7811222124625920000} g_5 \right. \\
 & \left. + \frac{576913638006109}{5680888817909760} g_1 + \frac{2198692458340927}{62489776997007360} g_4 + \frac{1712513569725097}{196921566167040000} g_0 \right) \\
 & + h^3 \left( \frac{9764298712220022539}{175752497804083200000} f_0 - \frac{974338363854221}{10414962832834560} f_4 + \frac{7479253404842309}{10414962832834560} f_1 \right. \\
 & \left. - \frac{1917077820047803}{3347666624839680} f_2 + \frac{4095180085353797}{6390999920148480} f_3 + \frac{102922176482877487}{58584165934694400000} f_5 \right) \\
 & hy'_1 + \frac{1}{3} y_0 - \frac{1}{7} y_1 - \frac{4}{21} y_2 = \\
 & + h^5 \left( \frac{273423769919}{26852940840960} Y_3 - \frac{7971534946763}{1621145558241973} Y_4 - \frac{14407022881829}{7030099912163328000} Y_5 \right. \\
 & \left. - \frac{223895422974491}{7030099912163328000} Y_0 - \frac{281203996486533120}{38573532399769} Y_1 - \frac{901294860533760}{6087105182556779} Y_2 \right) \\
 & + h^4 \left( \frac{7030099912163328000}{232557322391873} g_3 + \frac{281203996486533120}{29600420682792960} g_4 + \frac{901294860533760}{210902997364899840000} g_5 \right. \\
 & \left. - \frac{476083282786757}{772538451886080000} g_0 - \frac{13937786772754661}{562407992973066240} g_1 + \frac{155886621613211}{26362874670612480} g_2 \right) \\
 & + h^3 \left( \frac{8787624890204160}{476083282786757} f_3 + \frac{29600420682792960}{13937786772754661} f_4 + \frac{210902997364899840000}{155886621613211} f_5 \right. \\
 & \left. - \frac{772538451886080000}{13038057819361607} f_0 - \frac{562407992973066240}{3298316292306367} f_1 + \frac{26362874670612480}{16106755179979561} f_2 \right) \\
 & + h^2 \left( \frac{30128999623557120}{96094493918879089} f_2 - \frac{12406058668523520}{164838087583734911} f_3 - \frac{843611989459599360}{58592587599832309} f_4 \right. \\
 & \left. - \frac{27750394390118400000}{164838087583734911} f_0 - \frac{843611989459599360}{527257493412249600000} f_1 - \frac{58592587599832309}{527257493412249600000} f_5 \right) \\
 & hy'_2 + \frac{1}{3} y_0 - \frac{1}{7} y_1 - \frac{4}{21} y_2 = \\
 & + h^5 \left( \frac{273423769919}{26852940840960} Y_3 - \frac{7971534946763}{1621145558241973} Y_4 - \frac{14407022881829}{7030099912163328000} Y_5 \right. \\
 & \left. - \frac{223895422974491}{7030099912163328000} Y_0 - \frac{281203996486533120}{38573532399769} Y_1 - \frac{901294860533760}{6087105182556779} Y_2 \right) \\
 & + h^4 \left( \frac{7030099912163328000}{232557322391873} g_3 + \frac{281203996486533120}{29600420682792960} g_4 + \frac{901294860533760}{210902997364899840000} g_5 \right. \\
 & \left. - \frac{476083282786757}{772538451886080000} g_0 - \frac{13937786772754661}{562407992973066240} g_1 + \frac{155886621613211}{26362874670612480} g_2 \right) \\
 & + h^3 \left( \frac{8787624890204160}{476083282786757} f_3 + \frac{29600420682792960}{13937786772754661} f_4 + \frac{210902997364899840000}{155886621613211} f_5 \right. \\
 & \left. - \frac{772538451886080000}{13038057819361607} f_0 - \frac{562407992973066240}{3298316292306367} f_1 + \frac{26362874670612480}{16106755179979561} f_2 \right) \\
 & + h^2 \left( \frac{30128999623557120}{96094493918879089} f_2 - \frac{12406058668523520}{164838087583734911} f_3 - \frac{843611989459599360}{58592587599832309} f_4 \right. \\
 & \left. - \frac{27750394390118400000}{164838087583734911} f_0 - \frac{843611989459599360}{527257493412249600000} f_1 - \frac{58592587599832309}{527257493412249600000} f_5 \right) \\
 & hy'_3 - \frac{1}{9} y_0 + \frac{3}{7} y_1 - \frac{20}{63} y_2 = \\
 & + h^5 \left( \frac{18566034309583}{319549960074240} Y_3 - \frac{81739180753151}{93734665495511040} Y_4 + \frac{776360666563}{468673327477555200} Y_5 \right. \\
 & \left. + \frac{15740376465911}{1406019982432665600} Y_0 + \frac{9632539153807}{5513803852677120} Y_1 - \frac{15539847059497}{11716833186938880} Y_2 \right) \\
 & + h^4 \left( \frac{1406019982432665600}{285930654223} g_3 + \frac{1510555771010639}{187469330991022080} g_4 - \frac{110858657862671}{4686733274775552000} g_5 \right. \\
 & \left. + \frac{65093517705216}{429619603419221} g_0 + \frac{187469330991022080}{4877118247081} g_1 - \frac{4686733274775552000}{3142951770961} g_2 \right) \\
 & + h^3 \left( \frac{2008599974903808000}{127633153353227} f_2 - \frac{758985145712640}{52287648321183097} f_3 - \frac{266291663339520}{137620370469719} f_4 \right. \\
 & \left. - \frac{10042999874519040}{125275917298621151} f_0 + \frac{210902997364899840}{1806034298725621} f_1 + \frac{2840444408954880}{12971900645989} f_5 \right) \\
 & + \frac{105451498682449920000}{31244888498503680} f_0 + \frac{1806034298725621}{142309714821120000} f_1 + \frac{2840444408954880}{142309714821120000} f_5 \\
 & hy'_4 - \frac{5}{9} y_0 + y_1 - \frac{4}{9} y_2 = \\
 & + h^5 \left( \frac{772516240152409}{35150499560816640} Y_3 + \frac{9682247400401}{281203996486533120} Y_4 + \frac{3991497705557}{234336663738776000} Y_5 \right. \\
 & \left. + \frac{28810556220293}{540776916320256000} Y_0 + \frac{2641584488933807}{281203996486533120} Y_1 + \frac{14239037590031}{3905611062312960} Y_2 \right) \\
 & + h^4 \left( \frac{540776916320256000}{996895364794169} g_3 - \frac{2284794417673447}{2284794417673447} g_4 - \frac{1438856862344507}{1438856862344507} g_5 \right. \\
 & \left. + \frac{996895364794169}{2822650248295307} g_0 + \frac{26362874670612480}{64968790148319677} g_1 - \frac{70300999121633280000}{83997585921131} g_2 \right) \\
 & + \frac{2822650248295307}{2738999965777920000} g_0 + \frac{337444795783839744}{1687223978919198720} g_1 - \frac{83997585921131}{8787624890204160} g_2
 \end{aligned}$$

$$\begin{aligned}
 & +h^3 \left( \frac{5019402832618583}{10042999874519040} f_2 + \frac{648324158499559}{1127823515320320} f_3 + \frac{1015231427643113}{44400631024189440} f_4 \right. \\
 & \left. + \frac{179492540101149577}{31015146671308800000} f_0 + \frac{264315586188961837}{843611989459599360} f_1 + \frac{11145034417167397}{175752497804083200000} f_5 \right) \\
 & h y'_5 - y_0 + \frac{11}{7} y_1 - \frac{4}{7} y_2 = \\
 & +h^5 \left( -\frac{2474320358702441}{35150499560816640} \gamma_3 + \frac{15754608498784927}{281203996486533120} \gamma_4 + \frac{1286556248757211}{2343366637387776000} \gamma_5 \right. \\
 & \left. + \frac{196921566167040000}{777527017722287} \gamma_0 + \frac{21631076652810240}{217872672497189} \gamma_1 + \frac{35150499560816640}{3003908110663049} \gamma_2 \right. \\
 & \left. + h^4 \left( \frac{487506795164611}{8787624890204160} g_3 - \frac{8547862910467117}{51127999361187840} g_4 - \frac{2341374487672543423}{210902997364899840000} g_5 \right. \right. \\
 & \left. \left. + \frac{485348629286927}{196921566167040000} g_0 + \frac{709678353201473}{33082823116062720} g_1 + \frac{36871153385472}{100677581904186799} g_2 \right. \right. \\
 & \left. \left. + h^3 \left( \frac{30128999623557120}{6728928663328895447} f_2 + \frac{210902997364899840}{27970562413306063} f_3 + \frac{76691999041781760}{36284409313902219553} f_4 \right. \right. \right. \\
 & \left. \left. \left. + \frac{527257493412249600000}{64893229958430720} f_0 + \frac{27970562413306063}{64893229958430720} f_1 + \frac{36284409313902219553}{527257493412249600000} f_5 \right) \right) \\
 & h^2 y''_0 - \frac{4}{9} y_0 + \frac{4}{7} y_1 - \frac{8}{63} y_2 = \\
 & +h^5 \left( -\frac{16664893284479207}{140601998243266560} \gamma_1 - \frac{2628713142880333}{1506449981177856000} \gamma_0 - \frac{30509306352851}{270388458160128000} \gamma_5 \right. \\
 & \left. + \frac{5445193189306753}{17575249780408320} \gamma_2 - \frac{1527140229892357}{7532249905889280} \gamma_3 + \frac{2495083368365927}{56045572265460691} \gamma_4 \right) \\
 & +h^4 \left( \frac{1178182608722051}{4393812445102080} g_2 + \frac{272357513650643}{488201382789120} g_3 + \frac{56045572265460691}{35150499560816640000} g_5 \right. \\
 & \left. - \frac{19333862619163595}{56240799297306624} g_1 - \frac{34972927134020663}{281203996486533120} g_4 - \frac{4195652858855328007}{105451498682449920000} g_0 \right) \\
 & +h^3 \left( -\frac{37329944640834193093}{112983748588339200000} f_0 + \frac{15864607560202741}{46867332747755520} f_4 - \frac{93457679360213869}{46867332747755520} f_1 \right. \\
 & \left. + \frac{239495512055475461}{105451498682449920} f_2 - \frac{668252292421718447}{316354496047349760} f_3 - \frac{1617005711476367783}{263628746706124800000} f_5 \right) \\
 & h^2 y''_1 - \frac{4}{9} y_0 + \frac{4}{7} y_1 - \frac{8}{63} y_2 = \\
 & +h^5 \left( \frac{32541946004563}{3515049956081664} \gamma_3 - \frac{591875606331851}{421805994729799680} \gamma_4 + \frac{125083809920621}{17575249780408320000} \gamma_5 \right. \\
 & \left. + \frac{23907651160181}{450647430266880000} \gamma_0 + \frac{54761631104393}{421805994729799680} \gamma_1 - \frac{87932657577929}{2510749968629760} \gamma_2 \right) \\
 & +h^4 \left( -\frac{54761631104393}{1883062476472320} g_3 + \frac{9040264574624603}{843611989459599360} g_4 - \frac{2534247818977751}{25107499686297600000} g_5 \right. \\
 & \left. + \frac{517823638961607389}{527257493412249600000} g_0 - \frac{843611989459599360}{31805646704185573} g_1 + \frac{18026491567489}{1464604148367360} g_2 \right) \\
 & +h^3 \left( -\frac{65052977216754427}{105451498682449920} f_2 - \frac{1459352163911141}{105451498682449920} f_3 - \frac{51175565651977001}{1265417984189399040} f_4 \right. \\
 & \left. + \frac{993685385192999437}{188306247647232000000} f_0 - \frac{42472320831817139}{253083596837879808} f_1 + \frac{513093506305578941}{1318143733530624000000} f_5 \right) \\
 & h^2 y''_2 - \frac{4}{9} y_0 + \frac{4}{7} y_1 - \frac{8}{63} y_2 = \\
 & +h^5 \left( -\frac{19326393592417}{17575249780408320} \gamma_3 - \frac{87324170174617}{140601998243266560} \gamma_4 + \frac{138654971732807}{52725749341224960000} \gamma_5 \right. \\
 & \left. + \frac{11694886818157}{278972218736640000} \gamma_0 + \frac{217954784576261}{28120399648653312} \gamma_1 - \frac{1928800841713}{684749991444480} \gamma_2 \right) \\
 & +h^4 \left( \frac{69495574516033}{13181437335306240} g_3 + \frac{82951152289849}{16541411558031360} g_4 - \frac{1497557984757103}{40558268724019200000} g_5 \right. \\
 & \left. + \frac{39010187434770871}{47932499401113600000} g_0 + \frac{9307113402541321}{281203996486533120} g_1 - \frac{208917696311627}{2636287467061248} g_2 \right) \\
 & +h^3 \left( \frac{17539419487192847}{316354496047349760} f_2 - \frac{13503508567360997}{105451498682449920} f_3 - \frac{13366436187359}{551380385267712} f_4 \right. \\
 & \left. + \frac{863674520452631053}{188306247647232000000} f_0 + \frac{12128933222005907}{46867332747755520} f_1 + \frac{557198446400472887}{3954431200591872000000} f_5 \right) \\
 & h^2 y''_3 - \frac{4}{9} y_0 + \frac{4}{7} y_1 - \frac{8}{63} y_2 = \\
 & +h^5 \left( \frac{605978663485661}{52725749341224960} \gamma_3 - \frac{30438879798917}{28120399648653312} \gamma_4 + \frac{12120925130987}{2510749968629760000} \gamma_5 \right. \\
 & \left. + \frac{2326218064649593}{52725749341224960000} \gamma_0 + \frac{1024903694061337}{140601998243266560} \gamma_1 + \frac{171813393150337}{17575249780408320} \gamma_2 \right)
 \end{aligned}$$

$$\begin{aligned}
 &+h^4 \left( -\frac{2653172111453}{28717728399360} g_3 + \frac{2430632804338633}{8286650187130121} g_4 - \frac{12093095527656673}{175752497804083200000} g_5 \right. \\
 &+ \frac{445923094563607261}{527257493412249600000} g_0 + \frac{281203996486533120}{281203996486533120} g_1 + \frac{16180872807655}{878762489020416} g_2 \left. \right) \\
 &+h^3 \left( \frac{3511083949937077}{6203029334261760} f_2 + \frac{9308872019808853}{24334961234411520} f_3 - \frac{327625324403407}{9373466549551104} f_4 \right. \\
 &+ \frac{2662002133133893159}{564918742941696000000} f_0 + \frac{11626953675914387}{46867332747755520} f_1 + \frac{351349482944157629}{1318143733530624000000} f_5 \left. \right) \\
 &h^2 y''_4 - \frac{4}{9} y_0 + \frac{4}{7} y_1 - \frac{8}{63} y_2 = \\
 &+h^5 \left( \frac{768015602603423}{17575249780408320} \gamma_3 + \frac{1767231294914101}{421805994729799680} \gamma_4 - \frac{110774049786259}{17575249780408320000} \gamma_5 \right. \\
 &+ \frac{21107288955763}{532583326679040000} \gamma_0 + \frac{3404614177992011}{421805994729799680} \gamma_1 - \frac{157272776383}{270388458160128} \gamma_2 \left. \right) \\
 &+h^4 \left( -\frac{10979093491391}{13181437335306240} g_3 - \frac{576209763723007}{29669706369232667} g_4 + \frac{17477430198719263}{175752497804083200000} g_5 \right. \\
 &+ \frac{412172144166916829}{527257493412249600000} g_0 + \frac{843611989459599360}{111237895798323739} g_1 - \frac{1464604148367360}{495613382358785239} g_2 \left. \right) \\
 &+h^3 \left( \frac{9586499880222720}{840568231400999437} f_2 + \frac{105451498682449920}{66885468770335309} f_3 + \frac{1265417984189399040}{558726570238421059} f_4 \right. \\
 &+ \frac{188306247647232000000}{188306247647232000000} f_0 + \frac{253083596837879808}{253083596837879808} f_1 - \frac{1318143733530624000000}{1318143733530624000000} f_5 \left. \right) \\
 &h^2 y''_5 - \frac{4}{9} y_0 + \frac{4}{7} y_1 - \frac{8}{63} y_2 = \\
 &+h^5 \left( -\frac{5292706189748833}{17575249780408320} \gamma_3 + \frac{17602472808365927}{140601998243266560} \gamma_4 + \frac{2699138086776973}{1506449981177856000} \gamma_5 \right. \\
 &+ \frac{62327316853247}{390561106231296000} \gamma_0 - \frac{1557503844479207}{140601998243266560} \gamma_1 + \frac{11147442607920259}{52725749341224960} \gamma_2 \left. \right) \\
 &+h^4 \left( \frac{258563869889797}{1013956718100480} g_3 - \frac{103545793694020663}{802755615309085} g_4 - \frac{252278301678142231}{6203029334261760000} g_5 \right. \\
 &+ \frac{37316426638495999}{15064499811778560000} g_0 - \frac{8034399899615232}{8034399899615232} g_1 + \frac{1198312485027840}{684260944196923} g_2 \left. \right) \\
 &+h^3 \left( \frac{806807048166426383}{316354496047349760} f_2 - \frac{193310593473906149}{105451498682449920} f_3 + \frac{726912489232187}{327743585648640} f_4 \right. \\
 &+ \frac{24339663333604057}{2215367619379200000} f_0 - \frac{5373800960213869}{46867332747755520} f_1 + \frac{265147855161506896651}{790886240118374400000} f_5 \left. \right)
 \end{aligned}$$

### Analysis of the method

The analysis of the basic properties of the proposed method, which includes order, error constant, consistency, zero stability, and convergence were examined here.

### Local truncation error and order

In line with what has established in author [1], let the linear difference operator  $L$  associated with the CHMLMM be defined as:

$$L[y(x_n); h] = \sum_{j=0}^k (\alpha_j y(x_n + jh) - h^3 \beta_j y'''(x_n + jh) - h^4 \gamma_j y^{iv}(x_n + jh) - h^5 \mu_j y^v(x_n + jh)) \tag{11}$$

where  $y(x)$  is an arbitrary test function that is continuously differentiable in the interval  $[a,b]$ ,  $\alpha_j$ ,  $\beta_j$ ,  $\gamma_j$  and  $\mu_j$  are the continuous coefficients. Expanding  $(x_n + jh)$ ,  $y'''(x_n + jh)$ ,  $y^{iv}(x_n + jh)$  and  $y^v(x_n + jh)$  for  $j = 0, 1, \dots, m$  in Taylor's series about  $x_n$  and collecting the like terms in  $h$  and  $y$  gives

$$L[y(x); h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + C_3 h^3 y'''(x) + \dots + C_p h^p y^{(p)}(x) + \dots \tag{12}$$

The difference operator  $L$  and the associated hybrid multi-derivative linear multistep methods are said to be of order  $p$  if

$$C_0 = C_1 = C_2 = \dots = C_p = C_{p+1} = C_{p+2} = 0 \text{ and } C_{p+3} \neq 0 \text{ while the term } C_{p+3} \text{ is called the error constant and the local}$$

Truncation is given by  $\tau_k = C_{p+3} h^{(p+3)} y^{(p+3)}(x_n) + O(h^{(p+4)})$  Hence, the order and error constants associated with the methods developed are given below.

Table 1: Order and error CHMLMM

| Scheme | Order | Error Constant                     |
|--------|-------|------------------------------------|
| (8)    | 18    | 6713024908819                      |
|        |       | 542647169491165287284736000        |
| (9)    | 18    | 4356389612089                      |
|        |       | 813970754236747930927104000        |
| (10)   | 18    | 10368353484101                     |
|        |       | 348844608958606256111616000        |
|        | 18    | 11122508190817                     |
|        | 18    | 138128370415932982217932800        |
|        | 18    | 20760734273734313                  |
|        | 18    | 48838245254204875855626240000      |
|        | 18    | 288433693292887                    |
|        | 18    | 48838245254204875855626240000      |
|        | 18    | 46601507610661                     |
|        | 18    | 16279415084734958618542080000      |
|        | 18    | 452675887577653                    |
|        | 18    | 48838245254204875855626240000      |
|        | 18    | 867684662552429                    |
|        | 18    | 6976892179172125122232320000       |
|        | 18    | 164507915428536576763              |
|        | 18    | 219772103643921941350318080000     |
|        | 18    | 387788930067683                    |
|        | 18    | 24419122627102437927813120000      |
|        | 18    | 142511235164528583588138661        |
|        | 18    | 4050839414364769222969062850560000 |
|        | 18    | 466626111126631043869              |
|        | 18    | 24419122627102437927813120000      |
|        | 18    | 23648221493558910239419            |
|        | 18    | 3488446089586062561116160000       |
|        | 18    | 167621860351828513410431451731     |
|        | 18    | 75015544710458689314241904640000   |

### Consistency

CHMLMM is said to be consistent if the following conditions are satisfied according to [13] :

1. the order  $\rho \geq 1$
2.  $\sum_{j=0}^k \alpha_j = 0$
3.  $\rho(1) = \rho'(1) = 0$
4.  $\rho'''(1) = 3! \sigma(1)$

The consistency of CHMLMM examine as follows;

(i) the order of CHMLMM is  $p = 18$

(ii) The  $\alpha$ 's are;  $\alpha_0 = -\frac{4}{9}$ ;  $\alpha_1 = \frac{5}{7}$ ;  $\alpha_2 = -\frac{80}{63}$ ;  $\alpha_3 = 1$

$$\sum_{j=0}^k \alpha_j = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$$

$$-\frac{4}{9} + \frac{5}{7} - \frac{80}{63} + 1 = 0$$

(iii) Also,  $\rho(r)$  is the first characteristic polynomial here

$$\rho(r) = r^5 - \frac{80}{63}r^2 + \frac{5}{7}r - \frac{4}{9} = 0, \text{ when } r = 1$$

$$\rho'(r) = 5r^4 - \frac{40}{7}r^2 + \frac{5}{7}$$

$$\rho'(1) = 5r^4 - \frac{40}{7}r^2 + \frac{5}{7} = 0, \text{ when } r = 1$$

(iv) Similarly,  $\rho'''(1) = 60r^2 - 25r^{\frac{3}{2}} = 35$ , when  $r = 1$ ,  $3! \sigma(1) = 6$ , hence,  $\rho'''(1) = 3! \sigma(1)$   
Thus, conditions (1) - (4) are satisfied. This implies that CHMLMM is consistent.

### Zero Stability

**Definition 3.3.1:** The linear multistep method is said to be zero stable if no root of the first characteristic polynomial  $\rho(r)$  has a modulus greater than one and if every root of modulus one has multiplicity not more significant than three (see [14]).

$$\bar{\rho}(r) = \det[r\bar{A} - \bar{E}] \tag{13}$$

satisfies  $|r_s| \leq 1$  and every root with  $|r_s| = 1$  has multiplicity not exceeding three in the limit as  $h \rightarrow 0$ . The first characteristic polynomial for equation (8) is given by:

$$\rho(r) = r^5 - \frac{4}{9} + \frac{5r}{7} - \frac{80r^2}{63} \tag{14}$$

equating equation (15) to zero and solving for  $r$  gives

$$r^5 - \frac{4}{9} + \frac{5r}{7} - \frac{80r^2}{63} = 0$$

$$r = (1,1,1)$$

The root  $r$  of equation (15) for which  $|r| = 1$  is simple (since the multiplicity of the root  $r$  is three), hence the method for is zero stable as  $h \rightarrow 0$  by definition (see [13]).

**Numerical Examples**

In this section, we test some linear and nonlinear numerical examples to illustrate the accuracy of the methods. The maximum absolute error is computed as  $\max\text{ERR} = \max|y(x_i) - y_i|, i = 1, \dots, N$ , where  $y(x_i)$  is the exact solution computed at the grid point and  $y_i$  is an approximation to the exact solution using the CHMLMM. For each example, we find the absolute errors of the approximate solutions and were compared with various existing methods in the literature. The accuracy of our method is seen in the small error values obtained. The first test example considered is the linear third-order ODE

$$y''' = -y, \quad y(0) = 1, y'(0) = -1, y''(0) = 1, \quad 0 \leq x \leq 1, \quad h = 0.1$$

whose exact solution is  $y(x) = e^{-x}$ . The numerical solution was obtained in the interval [0,1] over ten iterations. The absolute errors of (CHMLMM) are presented in Table 2 as compared with those of [14].

Table 2: Comparison of absolute error of problem 1 using  $h = 0.1$

| $N$ | maxErr (CHMLMM) | Error in [14]  |
|-----|-----------------|----------------|
| .1  | $8.0E - 27$     | $2.8160E - 24$ |
| .2  | $4.0E - 27$     | $1.1025E - 23$ |
| .3  | $1.0E - 27$     | $2.4162E - 23$ |
| .4  | 0               | $1.797E - 23$  |
| .5  | $5.0E - 27$     | $6.3522E - 23$ |
| .6  | $2.0E - 27$     | $8.8946E - 23$ |
| .7  | $8.0E - 27$     | $1.1768E - 22$ |
| .8  | $2.0E - 27$     | $1.4936E - 22$ |
| .9  | $7.0E - 27$     | $1.8358E - 22$ |
|     | $5.0E - 27$     | $2.1997E - 22$ |

**Problem 2:** The second test example considered is the oscillatory problem

$$y''' = 3\sin x, \quad y(0) = 1; \quad y'(0) = 0; \quad y''(0) = -2; \quad h = 0.1$$

with the theoretical solution  $y(x) = 3\cos x + \frac{x^2}{2} - 2$ .

The proposed CHMLMM was applied to solve the second example in [0,1] over ten iterations and the absolute error maxERR are compared with those of [13] in the Table 3.

Table 3: Comparison of absolute error of problem 2 using  $h = 0.1$

| $N$ | maxErr (CHMLMM) | Error in [13]  |
|-----|-----------------|----------------|
| .1  | $4.0E - 27$     | $5.5511E - 17$ |
| .2  | 0               | $8.3266E - 17$ |
| .3  | $3.0E - 27$     | $5.5511E - 17$ |
| .4  | $2.0E - 27$     | $2.7755E - 16$ |
| .5  | $2.0E - 27$     | $2.2204E - 16$ |
| .6  | $4.0E - 27$     | $2.2204E - 16$ |
| .7  | $2.0E - 27$     | $6.6613E - 16$ |
| .8  | $8.0E - 27$     | $1.6653E - 15$ |
| .9  | $1.0E - 27$     |                |

**Problem 3:** We look at another problem

$$y''' = e^x, \quad y(0) = 3, \quad y'(0) = 1, \quad y''(0) = 5, \quad h = 0.1$$

$$\text{Exact: } y(x) = 2 + 2x^2 + e^x$$

We compare the results obtained with that of Duromola 2022

Table 4: showing the exact solution and computed results from the proposed method for the problem in example 2. With  $h = 0.1$

| Type equation here. | CHMLMM computed              | Exact                        |
|---------------------|------------------------------|------------------------------|
| .1                  | 3.12517091807564762481170783 | 3.12517091807564762481170783 |
| .2                  | 3.30140275816016983392107200 | 3.30140275816016983392107199 |
| .3                  | 3.52985880757600310398374431 | 3.52985880757600310398374431 |
| .4                  | 3.81182469764127031782485295 | 3.81182469764127031782485295 |
| .5                  | 4.14872127070012814684865079 | 4.14872127070012814684865079 |
| .6                  | 4.54211880039050897487536767 | 4.54211880039050897487536767 |
| .7                  | 4.99375270747047652162454939 | 4.99375270747047652162454939 |
| .8                  | 5.50554092849246760457953754 | 5.50554092849246760457953753 |
| .9                  | 6.07960311115694966380012657 | 6.07960311115694966380012656 |
|                     | 6.71828182845904523536028748 | 6.71828182845904523536028747 |

Table 5: Comparing the error obtained from the proposed method with that of Duromola 2022 for the problem in example 3. With  $h = 0.1$

| N  | maxErr (CHMLMM) | Error in Duromola (2022) |
|----|-----------------|--------------------------|
| .1 | 0               | $3.0834E - 24$           |
| .2 | $1.0E - 26$     | $1.2625E - 23$           |
| .3 | 0               | $2.9242E23$              |
| .4 | 0               | $5.3616E - 23$           |
| .5 | 0               | $8.6502E - 23$           |
| .6 | 0               | $1.2873E - 22$           |
| .7 | 0               | $1.8122E - 22$           |
| .8 | $1.0E - 26$     | $2.4500E - 22$           |
| .9 | $1.0E - 26$     | $3.2119E - 22$           |
|    | $1.0E - 26$     | $4.1103E - 22$           |

**Problem 4:** Application to solve nonlinear Genesio equation as seen in [13]. The chaotic Genesio equation is given as:

$$y''' = -\alpha y'' - \beta y' + f(y(x))$$

where,  $f(y(x)) = -\gamma y(x) + y^2(x)$   $y(0) = 0.2, \quad y'(0) = -0.3, \quad y'' = 0.1, \quad x \in [a, b], \quad \alpha = 1.2, \quad \beta = 2.92$  and  $\gamma = 6$  are positive constants that satisfied  $\alpha\beta < \gamma$ . The solution of the Genesio equation is considered considered in  $[0,1]$  over ten iteration. The results are presented in Table 6

Table 6: Solution of Genesis equation

| $N$ | (CHMLMM)<br>Computed |
|-----|----------------------|
| .1  | 0.170608506658       |
| .2  | 0.142832819577       |
| .3  | 0.117190915427       |
| .4  | 0.094091015252       |
| .5  | 0.073833210434       |
| .6  | 0.056613781195       |
| .7  | 0.042528410528       |
| .8  | 0.031584812170       |
| .9  | 0.023709496437       |
| .0  | 0.018757581303       |

**Problem 5:** Application to a boundary layer problem

$$2y''' + yy'' = 0, \quad y(0) = y'(0) = 0; \quad y''(0) = 1$$

There is no known exact solution for the problem. The solution of the boundary layer problem is considered in [0,1] over ten iteration. The results are presented in (Table 7).

Table 7: Solution of Boundary layer problem

| $N$ | (CHMLMM)<br>Computed |
|-----|----------------------|
| .1  | 0.904919042631       |
| .2  | 0.819375765873       |
| .3  | 0.742967136690       |
| .4  | 0.675342902080       |
| .5  | 0.616196858085       |
| .6  | 0.565259837891       |
| .7  | 0.522292708799       |
| .8  | 0.487081569716       |
| .9  | 0.459433058893       |
| .0  | 0.439170230840       |

**Conclusion**

This work has developed a class of hybrid multi-derivative linear multistep method (CHMLMM) to solve third-order ordinary differential equations directly. The method is zero and  $p$ -stable, consistent and convergent. The numerical results obtained from solved problems show the method's efficiency and accuracy advantages over existing methods in literature. The results in Tables 2-7 showed that the method is better in accuracy and can compete well with others in the literature for solving similar problems.

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